BHASKARA CONTEST - FINAL - JUNIOR

Classes IX & X

AMTI - Saturday, 2nd November_2019.

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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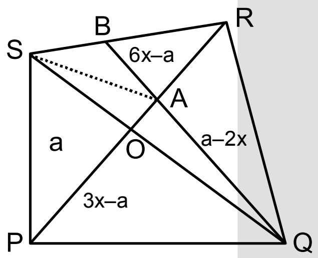
1. In a convex quadrilateral PQRS, the areas of triangles PQS, QRS and PQR are in the ratio 3 : 4 : 1. A line through Q cuts PR at A and RS at B such that PA : PR = RB : RS. Prove that A is the midpoint of PR and B is the midpoint of RS.

Sol.



Given $\frac{PA}{PR} = \frac{RB}{RS}$

Let $\frac{PA}{PR} = \frac{RB}{RS} = \frac{k}{k+1}$

ar(ΔSOR) × ar(POQ) = ar(ΔPOS) × ar(QOR)

(6x – a) (3x – a) = a(a – 2x)

18x = 7a

So, ar(ΔPOS) = a ; ar(ΔPOQ) = $\frac{a}{6}$; ar(ΔQOR) = $\frac{2a}{9}$; ar (ΔROS) = $\frac{4a}{3}$

Now, ar(ΔAOS) = $\left(\frac{k}{k+1}\right)\frac{7a}{3} - a = \left(\frac{4k-3}{3(k+1)}\right)a$

ar(ΔAOQ) = $\left(\frac{k}{k+1}\right)\frac{7a}{18} - \frac{a}{6} = \left(\frac{4k-3}{18(k+1)}\right)a$

ar(ΔARS) = $\left(\frac{1}{k+1}\right)\frac{7a}{3}$

ar(ΔBAS) = $\left(\frac{1}{k+1}\right)\left(\frac{1}{k+1}\right)\frac{7a}{3}$.

$\frac{ar(\Delta QSB)}{ar(\Delta QSR)} = \frac{1}{k+1} \Rightarrow ar(\Delta QSB) = \frac{1}{k+1}\, ar(\Delta QSR)$

ar(ΔAOS) + ar(ΔAOQ) + ar(ΔBAS) = $\frac{1}{k+1}$ ar(ΔQSR)

$$\left(\frac{4k-3}{3(k+1)}+\frac{4k-3}{18(k+1)}+\frac{7}{3(k+1)^2}\right)a = \frac{1}{k+1} \times \frac{14a}{9}$$

$$\frac{6(4k-3)(k+1)+(4k-3)(k+1))+42}{18(k+1)^2} = \frac{14}{9(k+1)}$$

$7(4k - 3)(k + 1) + 42 = 2 \times 14(k + 1)$

$(4k - 3)(k + 1) + 6 = 4(k + 1)$

$4k^2 + 4k - 3k - 3 + 6 = 4k + 4$

$4k^2 - 3k - 1 = 0$

$k = 1$

So, $\frac{PA}{AR} = \frac{k}{1} = \frac{1}{1}$

$\frac{RB}{BS} = \frac{k}{1} = \frac{1}{1}$.

$\therefore$ A is the midpoint of PR and B is the midpoint of SR.

2. Given positive real numbers a, b, c, d such that cd = 1. Prove that there exist at least one positive integer m such that $ab \le m^2 \le (a + c)(b + d)$.

Sol. $(bc - ad^2) \ge 0$

$b^2c^2 + a^2d^2 - 2abcd \ge 0$

add 4abcd on both sides

$b^2c^2 + a^2d^2 + 2abcd \ge 4abcd$

$(bc + ad)^2 \ge 4ab$ {cd = 1}

$(bc + ad) \ge 2\sqrt{ab}$

add both sides 1 + ab

$bc + ad + ab + 1 \ge 2\sqrt{ab} + ab + 1$

$bc + ab + ad + cd \ge 1 + ab + 2\sqrt{ab}$

$(a + c)(b + d) \ge \left(\sqrt{ab}+1\right)^2$

$\sqrt{(a+c)(b+d)} \ge \sqrt{ab}+1$

$\sqrt{(a+c)(b+d)} - \sqrt{ab} \ge 1$

let $\sqrt{(a+c)(b+d)} = k$, $\sqrt{ab} = \ell$

$k - \ell \ge 1$

$k \ge 1 + \ell$

$\ell^2 \le (\ell + 1)^2 \le k^2$

$ab \le m^2 \le (a + c)(b + d)$.

3. Find the number of permutations $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ of the integers – 3, – 2, 1, 0, 1, 2, 3, 4 that satisfy the chain of inequalities.

$x_1 x_2 \le x_2 x_3 \le x_3 x_4 \le x_4 x_5 \le x_5 x_6 \le x_6 x_7 \le x_7 x_8$.

Sol. We have 4 positive numbers 3 negative numbers and 0 one time.

Now,

$(+) \times (-) = (-)$

$(+) \times (+) = (+)$

$(+/-) \times 0 = 0$

No negative number can be placed right of 0

Also 2 consecutive numbers are not negative in the left of 0.

The sequence either start with positive or negative.

The positive numbers sequence should be descending order and the negative numbers sequence must be in ascending order.

The following sequence are possible

Case-1 : + – + – + = + 0 $\rightarrow$ Number of ways = 1

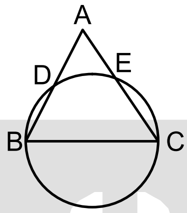
Case -2 : + – + – + – 0 + $\rightarrow$ Number of ways = ${}^4C_1$ = 4

Case-3 : – + – + – 0 + + $\rightarrow$ Number of ways = ${}^4C_2$ × 2 = 12

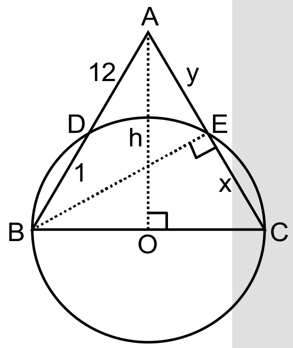
Case-4 : – + – + – + 0 + $\rightarrow$ Number of ways = ${}^4C_1$ = 4.

Total number of permutations = 1 + 4 + 12 + 4 = 21.

4. In the figure, BC is a diameter of the circle, where BC = $\sqrt{257}$, BD = 1, and DA = 12. Find the length of EC and hence find the length of the altitude from A to BC.



Sol.



AD = 12, BD = 1, BC = $\sqrt{257}$

Let AE = y, EC = x

AD × AB = AE × AC

12 × 13 = y (x + y)

$y^2 + xy = 156$ ....(1)

$BE^2 = 13^2 + y^2 = 257 - x^2$

$x^2 - y^2 = 88$ ....(2)

from equation (1) and (2)

y(x + y) = 156

(x – y) (x + y) = 88

$\Rightarrow \quad \frac{y}{x-y} = \frac{156}{88} = \frac{39}{22}$

$\Rightarrow \quad 22y = 39x - 39y$

$61y = 39x$

from equation (2)

$x^2 - \left(\frac{39}{61}x\right)^2 = 88 \quad \Rightarrow \quad \frac{2200}{3721}x^2 = 88$

$\frac{x}{61} = \frac{2}{10} = \frac{1}{5}$

$x = \frac{61}{5}$; EC = $\frac{61}{5}$

$y = \frac{39}{61} \times \frac{61}{5} = \frac{39}{5}$

Area of $\Delta ABC = BC \times h = AC \times BE$

$\sqrt{257} \times h = (x + y) \times BE$

$\sqrt{257} \times h = \left(\frac{61}{5} + \frac{39}{5}\right) \times \sqrt{169 - \left(\frac{39}{5}\right)^2}$

$\sqrt{257} \times h = 20 \times \sqrt{\frac{2704}{25}}$

$\sqrt{257} \times h = 20 \times \frac{52}{5}$

$h = \frac{208}{\sqrt{257}}$.

5. A math contest consists of 9 objective type questions and 6 fill in the blanks questions. From a school some number of students took the test and it was noticed that all students had attempted exactly 14 out of the 15 questions. Let $O_1, O_2, ..., O_9$ be the nine objective questions and $F_1, F_2, ..., F_6$ be the six fill in the blanks questions. Let a_{ij} be the number of students who attempted both questions O_i and F_j. If the sum of all the a_{ij} i = 1,2,3,......, 9 and j = 1,2,3,......., 6 is 972, then find the number of students who took the test in the school.

Sol. Let there are total 'n' students

Let there are 'p' student who attempt 8 objective and 6 fill up. {by 8 objective and 6 fill up there are 8 × 6 = 48 different as a_{ij} is possible}

Let there are 'q' student who attempt 9 objective and 5 fill up. {by 9 objective and 5 fill up there are 9 × 5 = 45 different as a_{ij} is possible}

As given in question sum of all a_{ij} is 972

$\therefore$ 48p + 45q = 972

16p + 15q = 324

16p = 324 – 15q

16p = 3(108 – 5q)___(1)

as RHS in multiple of 3

$\therefore$ LHS i.e., 16p is also multiple of 3

$\therefore$ p is multiple of 3.

let p = 3k

$\therefore$ eq. (1) became

16k = 108 – 5q

$q = \frac{108 - 16k}{5}$

by putting k = 3 we get p = 9 and q = 12

but if we put k = 8 or higher values of k, we can't get positive integral value of p and q

$\therefore$ total number of students n = p + q = 9 + 12 = 21

6. Find all positive integer triples (x, y, z) that satisfy the equation $x^4 + y^4 + z^4 = 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - 63$.

Sol. $x^4 + y^4 + z^4 - 2x^2y^2 + 2z^2x^2 - 2y^2z^2 = -63$.

$(x^2)^2 + (-y^2)^2 + (-z^2)^2 - 2x^2y^2 + 2y^2z^2 - 2y^2z^2 = 4y^2z^2 - 63$

$(x^2 - y^2 - z^2)^2 = 4y^2z^2 - 63$

$63 = 4y^2z^2 - (x^2 - y^2 - z^2)^2$

$63 = (2yz)^2 - (x^2 - y^2 - z^2)^2$

$63 = (2yz + x^2 - y^2 - z^2)(2yz - (x^2 - y^2 - z^2))$

$63 = (2yz - y^2 - z^2 + x^2)(2yz - x^2 + y^2 + z^2)$

$63 = (x^2 - (y - z)^2)((y + z)^2 - x^2)$

$63 = (x + y - z)(x - y + z)(y + z + x)(y + z - x)$

Now,

$63 = 63 \times 1 \times 1 \times 1$ ....(i)

$63 = 21 \times 3 \times 1 \times 1$ ....(ii)

$63 = 9 \times 7 \times 1 \times 1$ ....(iii)

$63 = 3 \times 3 \times 7 \times 1$ ....(iv)

again now, from (iv)

Let $x + y + z = 7$

$x + z - y = 3$ ....(A)

$x + y - z = 3$ ....(B)

$y + z - x = 1$ ....(C)

add (A) (B) (C) we get

$x + y + z = 7$

$x + y - z = 3$

(–) (–) (+) –

$2z = 4$

$z = 2,\ x = 2,\ y = 3$

again because of symmetry solution are

(2, 2, 3), (3, 2, 2,), (2, 3, 2)

Total 3 + 3 = 6 solutions.

(i) and (ii) are rejected as we can't get positive integral solution from that. Now from (iii)

Let $x + y + z = 9$

$x + y - z = 7$ ....(1)

$x + z - y = 1$ ....(2)

$z + y - x = 1$ ....(3)

Add (1) (2) and (3)

$x + y + z = 9$ ....(4)

$x + y - z = 7$ ....(1)

– – (+) –

$2z = 2$

$z = 1$

$x = y = 4$

So (4, 4, 1) is one solution.

as the question is symmetric in x, y, z so solution are (4, 4, 1) (1, 4, 4,) (4, 1, 4)

3 solutions.

7. The perimeter of ΔABC is 2 and its sides are BC = a, CA = b, AB = c. Prove that

$$abc + \frac{1}{27} \geq ab + bc + ca - 1 \geq abc.$$

Sol. As a + b + c = 2 (1 – a) (1 – b) (1 – c) all are positive.

$(1 - a)(1 - b)(1 - c) > 0$

$1 - (a + b + c) + ab + bc + ca - abc > 0$

$1 - 2 + ab + bc + ca - abc > 0$

$ab + bc + ca - abc - 1 > 0$

$ab + bc + ca - 1 > abc$

AM $\geq$ GM

as $\frac{(a+b-c)(b+c-a)(c+a-b)}{3} \geq \sqrt[3]{(a+b-c)(b+c-a)(c+a-b)}$

$$\frac{(2-2c)+(2-2a)+(2-2b)}{3} \geq \sqrt[3]{(2-2c)(2-2a)(2-2b)}$$

$$2\left[\frac{3-(a+b+c)}{3}\right] \geq \sqrt[3]{8\left[1-(a+b+c)+ab+bc+ca-abc\right]}$$

$$2 \times \frac{1}{3} \geq \sqrt[3]{8(1-2+ab+bc+ca-abc)}$$

$$\frac{8}{27} \geq 8(-1 + ab + bc + ca - abc)$$

$$\frac{1}{27} \geq ab + bc + ca - abc - 1$$

$$abc + \frac{1}{27} \geq ab + bc + ca - 1$$

Hence $abc + \frac{1}{27} \geq ab + bc + ca - 1 \geq abc$.

8. A circular disc is divided into 12 equal sectors and one of 6 different colours is used to colour each sector. No two adjacent sectors can have the same colour. Find the number of such distinct colourings possible.

Sol. Let f(n) be the number of valid ways to colour a circular disc with n sector (which we call an n-ring), so the answer is given by f(12). For n = 2, we compute $f(n) = 6 \cdot 5 = 30$. For $n \geq 3$, we can count the number of valid colourings a follows : choose one of the sector arbitrarily, which we may colour in 6 ways. Moving clockwise around the ring, there are 5 ways to colour each of the n – 1 other sector. Therefore, we have $6 \cdot 5^{n-1}$ colouring of an n-ring.

however, note that the first and last sector could be the same colour under this method. To count these invalid colourings, we see that by "merging " the first and last sections into one, we get a valid colouring of an (n – 1) ring. That is, there are f(n – 1) colourings of an n-ring in which the first and last sectors have the same colour. Thus,

$f(n) = 6 \cdot 5^{n-1} - f(n-1)$ for all $n \geq 3$.

To compute the requested value f(12), we repeated apply this formula.

$f(12) = 6.5^{11} - f(11)$

$= 6.5^{11} - [6.5^{10} - f(10)]$

$= 6.5^{11} - [6.5^{10} - \{6.5^{9} - f(9)]$

$= 6.5^{11} - 6.5^{10} + 6.5^{9} - f(9)$

and so an we get

$= 6[5^{11} - 5^{10} + 5^9 \ldots\ldots + 5]$

$$= 6\left[\frac{5^{11}\left(1-\left(\frac{-1}{5}\right)\right)^{11}}{1-\left(-\frac{1}{5}\right)}\right] = 6\left[\frac{5^{11}\left(\frac{5^{11}+1}{5^{11}}\right)}{\frac{6}{5}}\right] = 5(5^{11} + 1).$$