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REGIONAL MATHEMATICAL OLYMPIAD
2016

TEST PAPER WITH SOLUTION & ANSWER KEY

REGION: MAHARASTRA | CENTRE: GOA

Date: 09th October, 2016 | Duration: 3 Hours | Max. Marks: 100

:: IMPORTANT INSTRUCTIONS ::

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 100.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Find distinct positive integers $n_1 < n_2 < ..... < n_7$ with the least possible sum, such that their product
$n_1 \times n_2 \times ......\times n_7$ is divisible by 2016.

Sol. The choice of distinct positive integers n_1, n_2, ..., n_7 with the smallest possible sum is (1,2,...., 7), with sum 1 + 2 + ... + 7 = 28. However, we see that $1 \times 2 \times .... \times 7 = 5040 = 2016 \times \frac{5}{2}$. This means the above product is not divisible by 2016; it is falling short by just one factor of 2.

Hence, we will have to choose at least one number in our set, that is larger than 7. The smallest such choice available is 8; which will provide us with the required additional factor of 2.

Now, to decide which number to replace : We cannot replace 7 or 6, because both contain prime factors that are required for the product to be divisible by 2016. However, we observe that the number 5 is not needed, since 5 and 2016 are coprime. So we can remove 5 and put 8 in our set.

Hence we get the answer as (1, 2, 3, 4, 6, 7, 8); with sum 31; and product divisible by 2016.

Sol. We claim that the answer is (1, 2, 3, 4, 6, 7, 8); with sum 31; and product divisible by 2016.

To prove that this is the best choice, we simply list all the possibilities with sum less than 31:

(1,2,3,4,5,6,7) with sum 28; but the product does not have sufficient occurrences of the factor 2.

(1,2,3,4,5,6,8) with sum 29; but the product does not have sufficient occurrences of the factor 7.

(1,2,3,4,5,6,9) with sum 30; but the product does not have sufficient occurrences of the foctor 7.

(1,2,3,4,5,6,7,8) with sum 30; but the product does not have sufficient occurrences of the factor 3.

2. At an international event, there are 100 countries participating , each with its own distinct flag. There are 10 distinct flagpoles at the stadium, labeled # 1, #2, …., # 10 in a row. In how many ways can all the 100 flags be hoisted on these 10 flagpoles, such that for each i from 1 to 10, the flagpole # i has at least i flags?

(Note that the vertical order of flags on each flagpole is important.)

Sol. In order to distribute the flags on the poles, we use the following procedure :

(i) First we arrange all the 100 flags in a straight row; this can be done in 100! ways.

(ii) Now we have to decide the positions at which to break up the row into 10 separate segments, such that the first (left-most) segment goes onto flagpole ≠ 1, the next one onto pole ≠ 2, and so on.

Let us say that the number of flags on the poles ≠1 to ≠10 is given by $x_1, x_2, \ldots, x_{10}$ respectively.

So we have to find the number of integer solutions for the equation : $x_1 + x_2 + \ldots + x_{10} = 100$, such that $x_1 \geq 1$, $x_2 \geq 2$, etc. upto $x_{10} \geq 10$.

Let us define $y_1 = x_1 - 1$, $y_2 = x_2 - 2$, ..., $y_{10} = x_{10} - 10$. Then the above can be rephrased as :

$y_1 + y_2 + \ldots + y_{10} = 45$; with the constraint that $y_i \geq 0$ for i = 1 to 10.

This is a well-known standard problem, it is equivalent to choosing 45 objects from 10 categories, with repetitions allowed; or equivalent to arranging 45 identical objects and 9 identical separators' in a straight line. The same has $\binom{45+9}{9}$ solutions.

Finally, we observe that the steps (i) and (ii) mentioned at the beginning are idependent; i.e., for each of the 100! ways to arrange the flags in a straight row, we have $\binom{45+9}{9}$ distinct ways to split the row into 10 smaller segments, to get a different final configuration each time.

So by Multiplication Principle, the required answer is : $100! \times \binom{45+9}{9}$

3. Find all integers k such that all the roots of the following polynomial are also integers:

$$f(x) = x^3 - (k - 3)\, x^2 - 11x + (4k - 8).$$

Sol. suppose that for some value of k, all the roots of f(x) are integers. We observe that the coefficient of k in the expression of the polynomial is $(-x^2 + 4)$; meaning that for x = 2 and x = – 2, the value of the polynomial does not depend on k.

We get : f(–2) = 18 which is positive; and f(2) = – 10 which is negative. so at least one root lies between – 2 and 2.

Case 1 : One of the roots is –1. This implies f(–1) = 3k + 5 = 0; so $k = -\frac{5}{3}$, which is not an integer.

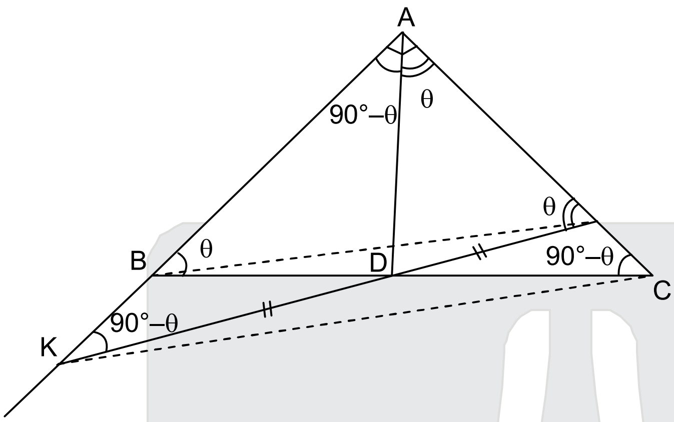
Case 2 : One of the roots is 0. This implies f(0) = 4k – 8 = 0; implying k = 2. In this case, the polynomial is : $f(x) = x^3 + x^2 - 11x = x(x^2 + x - 11)$. But the quadratic expression inside the bracket does not have integer roots.

Case 3 : One of the roots is 1. This implies f(1) = 3k – 15 = 0; impying k = 5. In this case, the polynomial is $f(x) = x^3 - 2x^2 - 11x + 12 = (x - 1)(x^2 - x - 12) = (x - 1)(x - 4)(x + 3)$. So that roots of the polynomial are 1, 4, – 3 which are all integers, as required.

Hence, the only solution is k = 5; giving $f(x) = x^3 - 2x^2 - 11x + 12$ with roots 1, 4 and – 3.

4. Let ΔABC be scalene, with BC as the largest side, Let D be the foot of the altitude from A onto side BC. Let points K and L be chosen on the lines AB and AC respectively, such that D is the midpoint of segment KL. Prove that the points B, K, C, L are concyclic if and only if $m\angle BAC = 90°$.

Sol.



Without loss of generality, let AB < AC. Therefore A – B – K and A – L –C.

First let us assume that $m\angle BAC = 90°$

Then in ΔAKL, D is the midpoint of hypotenuse KL; so D is in fact the circumcentre of ΔAKL; and ΔADK is isosceles. Hence we get,

$m\angle LKB = m\angle DKA = \angle DAK = m\angle DAB$

$= 90° - m\angle ABD = 90° - m\angle ABC = m\angle ACB = m\angle LCB$

Thus $m\angle LKB = m\angle LCB$; so the points B, K, L, C are concyclic.

Conversely, let us now assume that B, K, L, C are concyclic.

Let $m\ \angle ALK = x$. Then $m\angle KBC = m\angle KLC = (180º - x)$

so $m\angle ABD = 180º - m\angle KBC = x$

Hence $m\angle DAK = m\angle DAB = 90° - m\angle ABD = 90° - x$

Now if O is the circumcenter of ΔAKL, then ΔOAK is isosceles, with the central angle $m\angle AOK = 2(m\angle ALK) = 2x$; and side angle $m\angle OAK = 90° - x$

From (1) and (2), we see that $\angle DAK = \angle OAK$; meaning that O lies on line AD.

Also, O lies on the perpendicular bisector of seg KL which passes through D.

Hence O and D coincide; meaning that seg KL is in fact a diameter of the circumcircle of ΔAKL.

Hence $\angle KAL$ is inscribed in a semicircle, and equals 90°, as required.

5. Let x, y, z be non-negative real numbers such that xyz = 1. Prove that

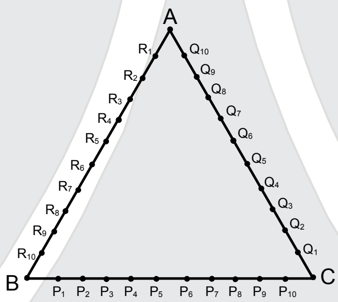
$$(x^3 + 2y)(y^3 + 2z)(z^3 + 2x) \geq 27$$

Sol. Applying AM-GM inequality to each of the brackets on the LHS, we get :

$$(x^3 + y + y)(y^3 + z + z)(z^3 + x + x) \geq (3\sqrt[3]{x^3y^2})(3\sqrt[3]{y^3z^2})(3\sqrt[3]{z^3x^2}) = 27$$

as required.

6. ABC is an equilateral triangle with side length 11 units. As shown in the figure, points P_1, P_2, …., P_{10} are taken on side BC in that order ; dividing the side into 11 segments of unit length each. Similarly, points Q_1 to Q_{10} are taken on side CA, and points R_1 to R_{10} are taken on side AB. Count the number of triangles of the form $P_iQ_jR_k$ such that their centroid coincides with the centroid of ΔABC. (Each of the indices I, j, k is chosen from {1, 2, ….., 10}, and need not be distinct.)



Sol. Let us setup a coordinate system in the plane, with B as the origin, and BC as the positive X-axis,

Hence the coordinates of the vertices are

$$B \equiv (0, 0);\ C \equiv (11, 0);\ A \equiv \left(\frac{11}{2}, \frac{11\sqrt{3}}{2\sqrt{3}}\right) \text{ and the centriod of } \Delta ABC \text{ is at } G\left(\frac{11}{2}, \frac{11}{2\sqrt{3}}\right)$$

Let P_i, Q_j, R_k be some points chosen on the sides of the triangle. By applying section formulas, we obtain the coordinates of these points along with their centroid G' as :

$$P_i \equiv (i, 0);\ Q_j \equiv \left(11 - \frac{j}{2}, \frac{j\sqrt{3}}{2}\right);\ RK \equiv \left(\frac{11-k}{2}, \frac{(11-k)\sqrt{3}}{2}\right);\ \text{and } G' \equiv \left(\frac{11}{2} + \frac{2i-j-k}{6}, \frac{11}{2\sqrt{3}} + \frac{j-k}{2\sqrt{3}}\right).$$

Comparing the coordinates of G and G', we see that they will coincide if and only if i = j = k.

Hence we have exactly 10 triangles, namely $\Delta P_1Q_1R_1$, $\Delta P_2Q_2R_2$, …….., $\Delta P_{10}Q_{10}R_{10}$, which satisfy the given condition.

Sol. Let P, Q, R be any points on segment BC, CA and AB respectively. We claim that the centriod of ΔPQR is the same as the centriod ΔABC if and only if BP = CQ = AR.

For the first part, let us assume that ΔABC and ΔPQR have a common centriod G.

Construction : Let D and S be the midpoints of segment BC and QR respectively. Extend AS to meet segment BC at point T.

Now, G divides the medians AD and PS in the ratio 2 : 1. Hence ΔDGS is similar to ΔAGP, and half its size. Therefore segment DS is parallel to and half the length of seg AP.

Hence, ΔXDS is similar to ΔXPA and half its size; so $XS = \frac{1}{2}XA$.

In other words, S is the midpoint of segment AX. But S is also the midpoint of QR

So this means ARXQ is a parallelogram; hence AR = XQ .... (1)

Also, AR||XQ implies $\angle CXQ = \angle CBA = \angle CAB = \angle XCQ$

So ΔXQC is isosceles, and XQ = CQ ....(2)

From (1) and (2), we get AR = CQ. By symmetry, we can prove that BP = CQ = AR, as required. Conversely, let us assume that BP = CQ = AR; and let G be the centriod of ΔPQR. We need to show that G is the centriod ΔABC.

Since ABC is equilateral, we also get CP = AQ = BR.

By SAS test, we see that triangles AQR, BRP and CPQ are congruent; so QR = RP = PQ; meaning ΔPQR is equilateral.

So G is not only the centriod but also circumcenter of PQR; so $\angle QGR = 120° = 180º - \angle QAR$

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