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# ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
# AMTI – NMTC – 2023 November – SUB-JUNIOR – FINAL

## INSTRUCTIONS

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before starting to answer, fill in the **FACE SLIP** completely.
5. Your 'rough work' should be done in the answer sheet itself.
6. Maximum time allowed is **THREE hours.**

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**1.** If $b(a^2 - bc)(1-ac) = a(b^2 - ca)(1-bc)$ where $a \neq b$ and $abc \neq 0$, prove that $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

**Sol.** $b(a^2 - bc)(1-ac) = a(b^2 - ca)(1-bc)$

$(a^2b-b^2c)(1-ac) = (ab^2-a^2c)(1-bc)$

$a^2b-b^2c-a^3bc + ab^2c^2 = ab^2-a^2c-b^3ac+a^2bc^2$

$a^2b-ab^2-b^2c + a^2c - a^3bc+ b^3ac+ab^2c^2-a^2bc^2=0$

$ab(a-b) + c(a^2-b^2)-abc(a^2-b^2) + abc(bc-ac) = 0$

$ab(a-b) + c(a^2-b^2)-abc(a^2-b^2) + abc^2(b-a) = 0$

$(a-b)(ab + c(a+b) - abc(a+b) - abc^2) = 0$

$(a-b)(ab + ac+ bc - a^2bc - ab^2c-abc^2) = 0$

$((ab+bc+ca) - abc(a+b+c)) = 0$

$ab + bc + ca = abc(a+b+c)$

$a+b+c = \frac{ab+bc+ca}{abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

**2.** a, b, c are three distinct positive integers. Show that among the numbers $a^5b-ab^5$, $b^5c-bc^5$, $c^5a-ca^5$, there must be one which is divisible by 8.

**Sol.** a b c $a^5b-ab^5$, $b^5c-bc^5$, $c^5a-ca^5$

(i) $\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}$

(i) $ab(a^4-b^4) = ab(a + b)(a - b)(a^2 + b^2)$ , $bc(b^4-c^4) = bc(b-c)(b+c)(b^2-c^2)$, $ca(c^4-a^4)=ac(c-a)(c+a)(c^2-a^2)$

(a) e × e × e

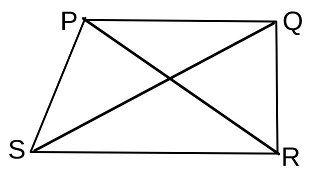
(ii) $\begin{bmatrix} E & E & E \\ E & 0 & E \\ 0 & E & 0 \end{bmatrix}$ (b) e × e × e

(iii) [E E E (iii) e × e × e × e

Hence all condition satisfy

**3.** There are four points P, Q, R, S on a plane such that no three of them are collinear. Can the triangles PQR, PQS, PRS and QRS be such that at least one has an interior angle less than or equal to 45° ? If so, how? If not, why?

**Sol.**




Case-1

When $\angle P = \angle Q = \angle R = \angle S = 90^\circ$

Then PQRS is a rectangle

So diagonal of rectangle are angle by bisectors

so each angle of triangle PQR PQS PRS and QRS = 45º

Case-2

When angle each is angle = P is not equal to 90º

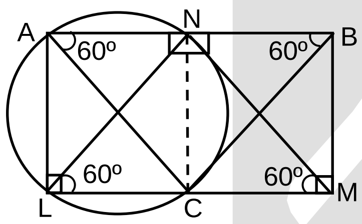
then at least one angle of a quadrilateral is less then 90º

then diagonal cuts the angle such that one angle > 45º angle other is < 45º.

So at least one interior angle of a triangle PQR PQS PRS and QRS less then for equal to 45º

**4.** A straight line $\ell$ is drawn through the vertex C of an equilateral triangle ABC, wholly lying outside the triangle. AL, BM are drawn perpendiculars to the straight line $\ell$. If N is the midpoint of AB, prove that $\Delta$LMN is an equilateral triangle.

**Sol.**




since N is the mid point of AB

AN = NB

IN $\Delta$ANC and $\Delta$BNC

AN = BN

NC = NC

AC = BC (equilateral $\Delta$ABC)

By SSS congruence criterion

$\Delta ANC \cong \Delta BNC$

By CPCT

$\angle ANC = \angle BNC$

So $\angle ANC = \angle BNC = 90^\circ$

Since IN Quadr. ANCL

$\angle ANC + \angle ALC = 180^\circ$

So Quadrilateral ANCL is cyclic Quadrilateral

By chord NC

$\angle NAC = \angle NLC = 60^o$

Similarly in quadrilateral BNMC

$\angle BNC + \angle CMB = 180^o$

BNCM $\rightarrow$ Cyclic Quadrilateral

By Chord NC

$\angle NBC = \angle NMC = 60^o$

In $\Delta NLM$

$\angle L = 60^o$

$\angle M = 60^o$

So $\angle N + \angle L + \angle M = 180^o$

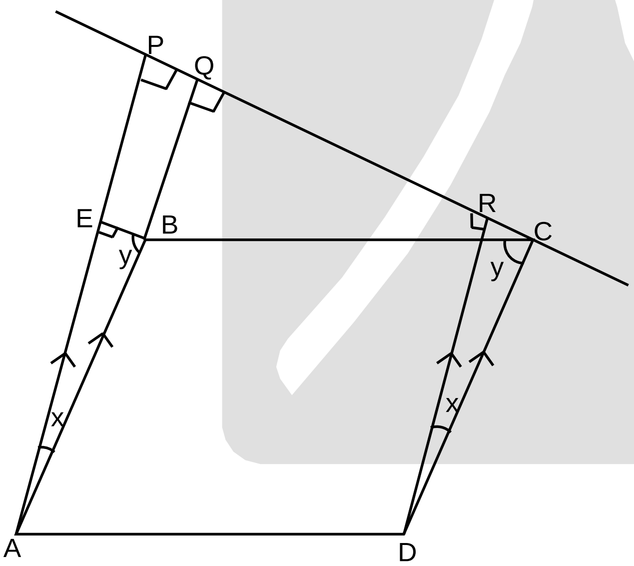
$\angle N + 60^o + 60^o = 180^o$

$\angle N = 60^o$

So $\Delta LMN \rightarrow$ Equilateral $\Delta$

**5.** ABCD is a parallelogram. Through C, a straight line is drawn outside the parallelogram. AP, BQ and DR are drawn perpendicular to this line. Show that AP = BQ + DR. If the line through C cuts one side internally, then will the same result hold? If so prove it. If not, what is the corresponding result? Justify your answer.

**Sol.** Case I:




Draw a line BE || QP

So, $\angle BEP = 90°$

$\because$ BE || PQ & EP || BQ & one angle in 90°

$\Rightarrow$ BEQP in rectangle

$\Rightarrow$ EP = BQ ........(i)

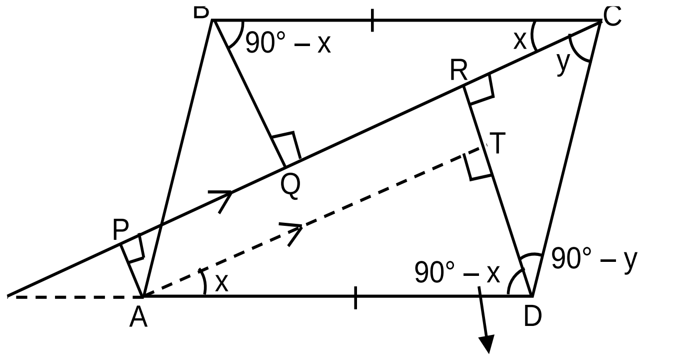
ABE $\cong$ $\Delta DCR$ {by ASA}

AE = DR ....... (ii) (by cpct)

Now, AP = AE + EP

AP = BQ + DR

Case II:




$\angle ADT = 180° - x - y - 90° + y$

Draw AT || PR

& $\angle P = 90°$

$\therefore$ APRT is a rectangle

$\Rightarrow$ AP = RT ............(i)

$\Delta ADT \cong \Delta CBQ$

DT = BQ ..........(ii)

DR = DT + TR

DR = BD + AP

AP = DR – BQ

**6.** m, n are non-negative real numbers whose sum is 1. Prove that the maximum and minimum values of $\dfrac{m^3+n^3}{m^2+n^2}$ are respectively 1 and $\dfrac{1}{2}$.

**Sol.** m + n = 1

for maximum value

$$\frac{m^3+n^3}{m^2+n^2} = \frac{(m+n)(m^2+n^2-mn)}{(m^2+n^2)} = \frac{(m^2+n^2-mn)}{m^2+n^2}$$

$$= \frac{(m+n)^2-3mn}{(m+n)^2-2mn}$$

$$= \frac{1-3mn}{1-2mn} \quad m, n \geq 0$$

for maximum value mn should be 0

so maximum value = 1

For minimum value

$$\frac{m^3+n^3}{m^2+n^2} = \frac{m^3+n^2-mn}{m^2+n^2} = 1-\frac{mn}{m^2+n^2} = 1-\frac{1}{\frac{m^2+n^2}{mn}}$$

$= 1 - \frac{1}{\frac{m}{n} + \frac{n}{m}}$ let $\frac{m}{n} = x$

$= 1 - \frac{1}{x + \frac{1}{x}}$

for minimum value $\frac{1}{x + \frac{1}{x}}$ should be maximum

for maximum value of $\frac{1}{x + \frac{1}{x}}$, $x + \frac{1}{x}$ should be minimum and $x + \frac{1}{x} \geq 2$

So minimum value of $x + \frac{1}{x} = 2$

Hence minimum value of $\frac{m^3 + n^2}{m^2 + n^2} = 1 = \frac{1}{2} = \frac{1}{2}$.

**7.** a) Solve for $x : \frac{x+5}{2018} + \frac{x+4}{2019} + \frac{x+3}{2020} + \frac{x+2}{2021} + \frac{x+1}{2022} + \frac{x}{2023} = -6$

b) If $\frac{a^2+b^2}{725} = \frac{b^2+c^2}{149} = \frac{c^2+a^2}{674}$ and a – c = 18 , find the value of (a + b + c).

**Sol.** **(a)** $\frac{x+5}{2018} + \frac{x+4}{2019} + \frac{x+3}{2020} + \frac{x+2}{2021} + \frac{x+1}{2022} + \frac{x}{2023} = 6$

$\frac{x+5}{2018} + 1 + \frac{x+4}{2019} + 1 + \frac{x+3}{2020} + 1 + \frac{x+2}{2021} + 1 + \frac{x+1}{2022} + 1 + \frac{x}{2023} + 1 = 0$

$\frac{x+2023}{2018} + \frac{x+2023}{2019} + \frac{x+2023}{2020} + \frac{x+2023}{2021} + \frac{x+2023}{2022} + \frac{x+2023}{2023} = 0$

$(x + 2023)\left(\frac{1}{2018} + \frac{1}{2019} + \frac{11}{2020} + \frac{1}{2021} + \frac{1}{2022} + \frac{1}{2023}\right) = 0$

$x + 2023 = 0$

$x = -2023$

**(b)** $\frac{a^2+b^2}{725} = \frac{b^2+c^2}{149} = \frac{c^2+a^2}{674}$, a – c = 18

Let $\frac{a^2+b^2}{725} - \frac{b^2+c^2}{149} = \frac{c^2+a^2}{674} = K$

$a^2 + b^2 = 725K$ … (i)

$b^2 + c^2 = 149K$ … (ii)

$c^2 + a^2 = 674K$ … (iii)

Adding above equation

$2(a^2 + b^2 + c^2) = 1548K$

$a^2 + b^2 + c^2 = 774K$ .. (iv)

byes$^n$ (1) & (iv)

$725K + c^2 = 774K$

$c^2 = 774K - 725K$

$c^2 = 49K$

by $eq^n$ (2) and (4)

$149K + a^2 = 774K$

$a^2 = 625K$

by $eq^n$ (iii) and (iv)

$674K + b^2 = 774K$

$b^2 = 100K$

Given $a – c = 18$

$25\sqrt{k} - 7\sqrt{k} = 18$

$18\sqrt{k} = 18$

$\sqrt{k} = 1$

$K = 1$

So, $a^2 = 625 \Rightarrow a = 25$

$b^2 = 100 \Rightarrow b = 10$

$c^2 = 49 \Rightarrow c = 7$

$a + b + c$

$25 + 10 + 7$

$42$

$a = -25$

$b = -10$

$c = -7$

$a - c = -18 \rightarrow$ Neglected

**8.** If $a + b + c + d = 0$, prove that $a^3 + b^3 + c^3 + d^3 = 3(abc + bcd + cda + dab)$.

**Sol.** $a + b + c + d = 0$

cubic on both sides.

$(a + b + c + d)^3 = 0$

$(a + b)^3 + (c + d)^3 + 3(a + b)(c + d)(a + b + c + d) = 0$

$(a + b)^3 + (c + d)^3 + 0 = 0$

$a^3 + b^3 + 3ab(a + b) + c^3 + d^3 + 3cd(c + d) = 0$

$a^3 + b^3 + c^3 + d^3 + 3(ab(a + b) + cd(c + d)) = 0$

$a^3 + b^3 + c^3 + d^3 + 3(ab(-c - d) + cd(-a - b)) = 0$

$a^3 + b^3 + c^3 + d^3 + 3(-abc - abd - acd - bcd) = 0$

$a^3 + b^3 + c^3 + d^3 + 3(abc + bcd + cda + dab) = 0$