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# NATIONAL BOARD FOR HIGHER MATHEMATICS
AND
# HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
# TATA INSTITUTE OF FUNDAMENTAL RESEARCH

# REGIONAL MATHEMATICAL OLYMPIAD, 2023
# (All Region)

# QUESTION PAPER WITH SOLUTION

**Sunday, October 29, 2023 | Time: 1 PM – 4 PM**

**Time : 3 hours (समय : 3 घंटे)** **October 29, 2023** **Total marks (अधिकतम अंक) : 102**

**Instruction निर्देश :**

**1.** Calculators (in any form) and protractors are not allowed.

कैलकुलेटर (किसी भी रूप में) या चांदा लाने की अनुमति नहीं है।

**2.** Rulers and compasses are allowed.

रूलर एवं प्रकार लाने की अनुमति है।

**3.** Answer all the questions. Draw neat Geometry diagrams.

सभी प्रश्नों के जवाब दें।

**4.** All questions carry equal marks. Maximum marks 102

सभी प्रश्न बराबर अंकों के हैं। अधिकतम अंक : 102

**5.** Answerer to each question should start on a new page, clearly indicate the question number.

हर प्रश्न का हल नए पन्ने से शुरू करें। प्रश्न संख्या का साफ–साफ उल्लेख करें।

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**1.** Let N be the set of all positive integers and S = $\{(a,b,c,d) \in \mathbb{N}^4 : a^2 + b^2 + c^2 = d^2\}$. Find the largest positive integer m such that m divides abcd for all $(a,b,c,d) \in S$.

**Sol.** **<u>1st method</u>**

Let a = n, b = n + 1, c = n (n + 1) $\Rightarrow$ d = (n (n + 1) + 1) which satisfy $a^2 + b^2 + c^2 = d^2$

a & b are two consecutive number in which one is odd & other is even c & d are two consecutive integers in which c is even & d is odd

$\Rightarrow$ abc is divisible by 4

If a = 1, b = 2, c =2, d = 3

abcd = 12 $\Rightarrow$ m $\le$ 12

now we chick abcd is divisible by 3 or not.

n = 3k, 3k + 1, 3k + 2

If n = 3k then abcd is divisible by.

If n = 3k + 1, d = (3k + 1) (3km+ 2) + 1 = 3k'

$\Rightarrow$ abcd is divisible 3.

If n = 3k + 2 then b = n + 1 = 3 (k + 1)

$\Rightarrow$ abcd is divisible by 12

$\Rightarrow$ largest positive integer m = 12

**<u>2st method</u>**

$a^2 \equiv 0, 1 \pmod 4$

$b^2 \equiv 0, 1 \pmod 4$

$c^2 \equiv 0, 1 \pmod 4$

$d^2 \equiv 0, 1 \pmod 4$

$\because a^2 + b^2 + c^2 = d^2$

$\because$ at least two of $a^2,b^2,c^2,d^2$ are even

$\therefore$ 4 divide abcd

now again $a^2 \equiv 0, 1$ (mode 3)

$b^2 \equiv 0, 1$ (mode 3)

$c^2 \equiv 0, 1$ (mode 3)

$b^2 \equiv 0, 1$ (mode 3)

$\because a^2 + b^2 + c^2 = d^2$

$\therefore$ at least on of $a^2, b^2, c^2, d^2$ is multiple of 3

$\therefore$ 3 divide abcd

$\Rightarrow$ 12 divide abcd

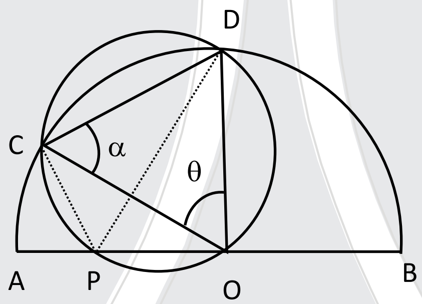
If a = 1, b = 2, c = 3 d = 3

So minimum value of abcd = 12

$\therefore$ m = 12

**2.** Let $\omega$ be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C,D lie in the interior of the are AB. Let E be in point on the diameter AB such that CE and DE are equally inclined to the line AB. Prove that
(a) the measure of $\angle$CED is a constant;
(b) the circumcircle of triangle CED passes through a fixed point.

**Sol.** Let CD = $\ell$




Now consider a circumcircle of $\Delta$CDO where 'O' is mid-point of diameter AB.

This circle intersect AB at P, now join P to C & D

$\angle$ COD = $\angle$ CPD = Angle on same segment = $\theta$ (Say)

$\angle$ OCD = $\angle$ OPD = Angle on same segment = $\alpha$

$\because$ OC = OD = radius of semi- circle

$\therefore \angle$ OCD = $\angle$ ODC = $\alpha$

$$\alpha + \alpha + \theta = \pi \Rightarrow \alpha = \frac{\pi}{2} - \frac{\theta}{2}$$

$\because \quad \angle APC + \angle CPD + \angle OPD = \pi$

$$\angle APC = \pi - \theta - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\theta}{2} = \angle OPD$$

$\therefore$ P is point E. $\Rightarrow$ Proof of Part (a)

and (b) circumcircle $\Delta$DCE passed though a fixed point O

**3.** Far any natural number n, expressed in base 10, let s(n) denote the sum of all its digits. Find all natural numbers m and n such that m < n and $(S(n))^2 = m$, $(S(m))^2 = n$

**Sol.** Since m < n and $(S(n))^2 = m$, $(S(m))^2 = n \Rightarrow$ m & n are perfect square

**Case – I** If n is one-digit number

max S(n) = 9 $\Rightarrow m \le 81 \Rightarrow$ S(m) = 9 $\Rightarrow n \le 81$

possible value of n = 1, 4, 9

| n | S(n) | m = $(S(n))^2$ | S(m) | n = $(Sm)^2$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 4 | 4 | 16 | 7 | 49 |
| 9 | 9 | 81 | 9 | 81 |

which is not possible

**Case –II** If n is of 2-digit number

max S(n) = 18 $\Rightarrow$ m = $(S(n))^2$ = 324.

S(m) = 9 $\Rightarrow$ $(S(m))^2$ = 81 $\Rightarrow$ n $\le$ 81

| n | S(n) | m = $(S(n))^2$ | S(m) | n = $(Sm)^2$ |
|---|---|---|---|---|
| 16 | 7 | 49 | 13 | 169 |
| 25 | 7 | 49 | 30 | 169 |
| 36 | 9 | 81 | 9 | 81 |
| 49 | 14 | 196 | 16 | 256 |
| 64 | 10 | 100 | 1 | 1 |
| 81 | 9 | 81 | 9 | 81 |

which is not possible

**Case –III** If n is of 3-digit number

max S(n) = 27 $\Rightarrow$ m = $(27)^2$ = 729

$(S(m))^2$ = 324 $\Rightarrow$ n $\le$ 324

| n | S(n) | m = $(S(n))^2$ | S(m) | n = $(Sm)^2$ |
|---|---|---|---|---|
| 100 | 1 | 1 | 1 | 1 |
| 121 | 4 | 16 | 7 | 49 |
| 144 | 9 | 81 | 9 | 81 |
| 225 | 9 | 81 | 9 | 81 |
| 256 | 13 | 169 | 16 | 256 |
| 289 | 19 | 361 | 10 | 100 |
| 324 | 9 | 81 | 9 | 81 |

n = 256, m = 169

**Case –IV** If n is of 4-digit

max (S(n)) = 36 $\Rightarrow$ m = $(S(n))^2$ = 1296

(S(n)) = 18 $\Rightarrow$ $(S(m))^2$ = 324 $\Rightarrow$ n $\le$ 324

not possible

**Case –V** n is of 5-digit

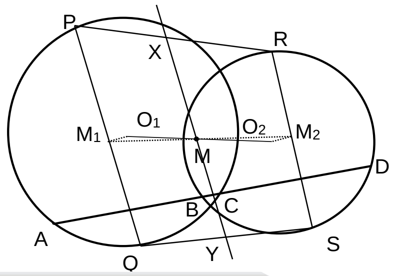
max (S(n)) = 45 $\Rightarrow$ (m) = 2025

S(m) = 9 $\Rightarrow$ $(S(m))^2$ = 81 $\Rightarrow$ n $\le$ 81

Not possible

So it can't have any solution for higher digits.

**4.** Let $\Omega_1, \Omega_3$ be two intersecting circles with centres $O_1$, $O_2$ respectively. Let $\ell$ be a line that intersects $\Omega_1$ at points A,C and $\Omega_2$ at points B,D such that A,B,C,D are collinear in that order. Let the perpendicular bisector of segment AB intersect $\Omega_1$ at point P,Q and the perpendicular bisector of segment CD intersect $\Omega_2$ at points R,S such that P,R are on the same side of $\ell$. Prove that the midpoints of PR,QS and $O_1O_2$ are collinear:




**Sol.** Let $M_1$ & $M_2$ be mid point of PQ and RS

$\therefore$ $O_1M_1 \parallel O_2M_2 \parallel \ell$

and $O_1M_1 = \frac{AC - AB}{2} = \frac{BC}{2}$ , $O_2M_2 = \frac{BD - CD}{2} = \frac{BC}{2}$

Now join $M_1$ and $M_2$ which interest $O_1O_2$ at M. Which is mid point of $O_1O_2$ and $M_1M_2$ both

Now drawn a line parallel to PQ through M which interest PR and QS at X and Y.
Since M is mid point of $M_1 M_2$ therefore X and Y are also mid-point PR and QS.
Hence proved.

**5.** Let $n > k > 1$ be positive integers. Determine all positive real numbers $a_1, a_2, \ldots, a_n$ which satisfy

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_i = \pi$$

**Sol.** Apply A.M. G.M on $(k-1)\ a_i^k$ and 1

$$\frac{a_i^k + a_i^k + \ldots\ldots\ldots + a_i^k + 1}{k} \geq \left(a_i^{k-1}\right)$$

$$\frac{(k-1)a_i^k + 1}{ka_i^k} \geq a_i^{-1}$$

$$\frac{ka_i^k}{(k-1)a_i^k + 1} \leq a_i$$

$$\Rightarrow \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sqrt{a_i}$$

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sum_{i=1}^{n}\sqrt{a_i} \qquad \ldots\ldots\ldots(1)$$

Janson's inequality on f(x) = $\sqrt{x}$

$$\frac{\sqrt{a_1} + \sqrt{a_2} + \ldots.. + \sqrt{a_n}}{n} \leq \sqrt{\frac{a_1 + a_2 + a_n}{n}}$$

$$\sum_{i=1}^{n}\sqrt{a_i} \leq \sqrt{n}.\sqrt{(\Sigma a_i)} \qquad \ldots\ldots\ldots(2)$$

but given $\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n}\sqrt{a_i} = n$ ………(3)

so from (1), (2) & (3)

$$n = \sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sqrt{n}.\sqrt{n} = n$$

which is possible only when equality in (1) and (2) both holds

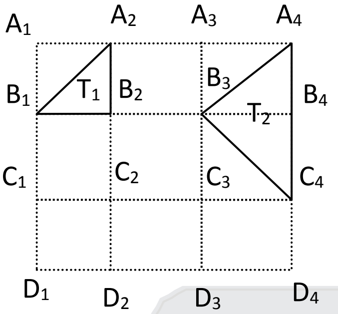
for equality in (2) $a_1 = a_2 = \ldots.. = a_n$

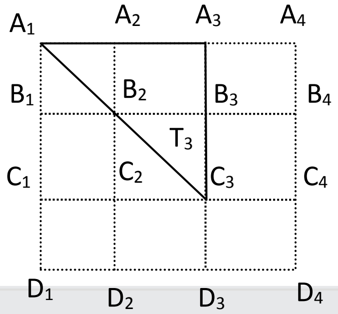
for equality in (1) $a_i^k = 1 \Rightarrow a_i = 1$

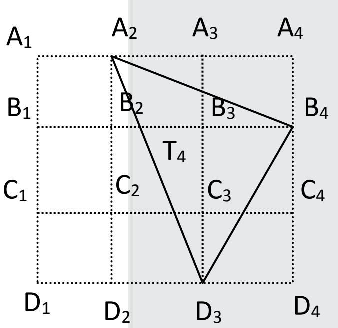
$\Rightarrow a_1 = a_2 = \ldots\ldots.. = a_n = 1$ is only

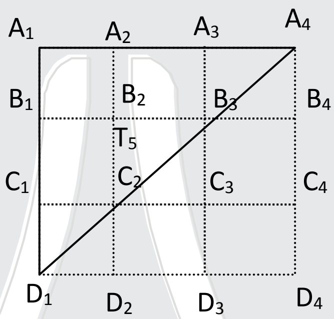
**6.** Consider a set of 16 points arranged in a 4 × 4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

**Sol.** Five types of isosceles right-angled triangle are possible as shown below



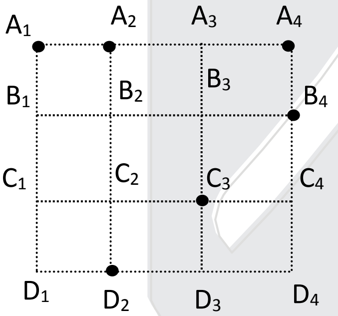




Now consider following cases

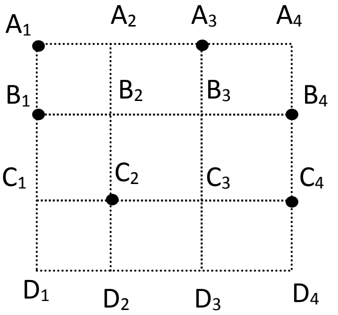
**Case-I:** If we take 4-points in one row and other 3-points scattered then we always get triangle of type $T_1$, $T_3$ or $T_5$

**Case-II:** If we take 3-points in one row and other 4-points scattered




Suppose three points taken are $A_1$, $A_2$, $A_4$ to not form isosceles right-angled triangle fourth, fifth and sixth points must be $B_4$,$C_3$ & $D_2$. Still one Point need to select, If we select any point from $R_2$, $R_3$ & $R_4$ then always we get isosceles right-angled triangle.

**Case-III:** If no row contain more than 2-points then 3 rows must contain 2-points each.




Suppose two points taken are $A_1$, $A_3$ to not form isosceles right-angled triangle third, fourth, fifth and sixth points taken $B_1$, $B_4$,$C_2$,$C_4$. Still one Point need to select, if we select any point from $R_4$, then always we get isosceles right-angled triangle.

If $D_1$ is selected $B_1C_2D_1$ is isosceles right-angled triangle.

If $D_2$ is selected $A_3B_1D_2$ is isosceles right-angled triangle.
If $D_3$ is selected $C_2D_3C_4$ is isosceles right-angled triangle.
If $D_4$ is selected $A_3C_2D_4$ is isosceles right-angled triangle.